Pythagorean triangles within Pythagorean triangles

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1 Introduction

Suppose that CBA is a Pythagorean triangle with sidelengths $|\overline{AB}| = c$, $|\overline{CA}| = b$, and $|\overline{CB}| = a$; that is, a right triangle with the right angle at C; and with a, b, c being positive integers such that $a^2 + b^2 = c^2$. Then (without loss of generality – a and b may be switched),

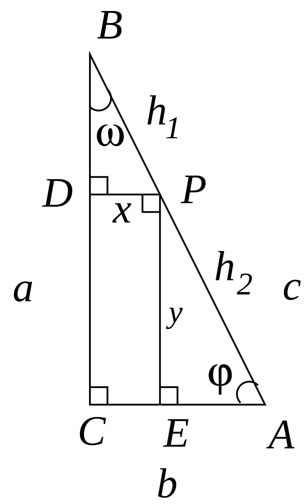

Figure 1

$$\left\{ \begin{array}{l} a = d(m^2 - n^2),\ b = d(2mn),\ c = d(m^2 + n^2) \\ \\ \text{where } d, m, n \text{ are positive integers such that} \\ \\ m > n,\ (m,n) = 1,\ \text{and } m + 1 \equiv 1(\text{mod } 2) \end{array} \right\} \tag{1}$$

Note: Throughout this paper, (X, Y) will stand for the greatest common divisor of two integers X and Y.

Thus, the condition $(m, n) = 1$ says that m and n are relatively prime, their greatest common divisor is 1. Also, the condition $m + n \equiv 1(\text{mod}2)$ says that m and n have different parities; one of them is even, the other odd. The formulas in (1), are the well known parametric formulas describing the entire family of Pythagorean triangles or triples.

A derivation of the formulas can be found in references [1] and [2]. For a wealth of historic information on Pythagorean triangles see [2] or [3].

Now, consider a point P on the hypotenuse $\overline{AB}$, and let D and E be the intersection points of the two lines through P and parallel to $\overline{CA}$ and $\overline{CB}$; with the sides $\overline{CB}$ and $\overline{CA}$ respectively. Two right triangles are formed; the triangles BDP and APE. Let x and y denote the lengths of line segments $\overline{DP}$ and $\overline{PE}$ respectively. Also, let $h_1 = |\overline{BP}|$ and $h_2 = |\overline{AP}|$. Then,

$$\left\{\begin{array}{l} \left|\overline{DP}\right| = \left|\overline{CE}\right| = x \text{ and } \left|\overline{PE}\right| = \left|\overline{DC}\right| = y. \\ \\ \text{Thus, } \left|\overline{BD}\right| = \left|\overline{BC}\right| - \left|\overline{DC}\right| = a - y; \\ \\ \text{and } \left|\overline{AE}\right| = \left|\overline{AC}\right| - \left|\overline{CE}\right| = b - x \end{array}\right\} \tag{2}$$

Both right triangles BDP and APE are similar to the right triangle of CBA. We have the similarity ratios,

$$\left\{\begin{array}{lllll} \dfrac{x}{b} & = & \dfrac{a-y}{a} & = & \dfrac{h_1}{c} \\ \\ \dfrac{y}{a} & = & \dfrac{b-x}{b} & = & \dfrac{h_2}{c} \end{array}\right\} \begin{array}{r} (3\text{i}) \\ \\ (3\text{ii}) \end{array}$$

Since a, b, c are (positive) integers, it follows, by inspection, from (3i) that if one of x, y, or h_1 is a rational number, then all three of them must be rational numbers. Hence, either all three x, y, h_1 are rationals, or otherwise, all three of them must be irrational. Likewise, it follows from (3ii) that either all three x, y, h_2 are rational or all three are irrational. Combining these two observations, we infer that

> *Either all four x, y, h_1, h_2 are rational numbers or, otherwise, all four of them are irrationals.*

In Section 2, we state three lemmas from number theory. One of them (Euclid's Lemma) is well known. We offer proofs for the other two.

In Section 3, we prove Theorems 1 and 2; Theorem 2 is a corollary of Theorem 1.

In Section 4,we consider and analyze three special cases. These are the cases when the point P is the midpoint M of the hypotenuse $\overline{AB}$; when P is the point I where the angle bisector of the $90°$ angle at C intersects the hypotenuse $\overline{AB}$, and when the point P is the foot F of the perpendicular from C to the hypotenuse $\overline{AB}$.

Back to Section 3. In Theorem 1 we prove that the two right triangles BDP and PEA in Figure 1 are either both Pythagorean or neither of them is a Pythagorean triangle (assuming, of course, that BCA is a Pythagorean triangle). It then follows, and this is part of Theorem 2, that when the triangle BCA is a primitive Pythagorean triangle, neither of the triangles,

BDP and PEA are Pythagorean for any position of the point P along the hypotenuse $\overline{AB}$.

In Section 5 (Theorem 6), we postulate that given a Pythagorean triangle with side lengths $a = d(m^2 - n^2),\ b = d(2mn)$, and $c = d(m^2 + n^2)$, where d, m, n are positive integers such that $d \geq 2$, $(m, n) = 1$, $m > n$, and $m + n \equiv 1(\text{mod } 2)$. Then there are exactly $d - 1$ positions of the point P, such that triangles BDP and PEA are both Pythagorean.

In Section 6, we will examine the general question of when, in addition to the two triangles BDP and APE being Pythagorean, the four congruent right triangles (within the rectangle $CDPE$) CDP, CEP, DCE, and EPD are also Pythagorean. We derive a family of non-primitive Pythagorean triangles CBA with that property.

Note: In addition to the notation (k, ℓ) denoting the greatest common divisor of two integers, k and ℓ, the notation $t|v$, will stand for "The integer t is a divisor of the integer v".

2 Three lemmas from number theory

Lemma 1. *(Euclid's Lemma): Suppose that a, b, c, are natural numbers such that $c|ab$ (i.e., c is a divisor of the product ab). If $(c, a) = 1$, then $c|b$.*

For a proof of this well-known result, the reader may refer to [1] or [2].

Lemma 2. *Let m, n be positive integers such that $m > n$, $(m, n) = 1$, and $m + n \equiv 1(\text{mod}2)$. Then*

(i) $(m^2 + n^2, 2mn) = 1$

(ii) $(m^2 + n^2, m^2 - n^2) = 1$

(iii) $(m^2 - n^2, 2mn) = 1$

Proof.

(i) We show that $m^2 + n^2$ and $2mn$ have no prime divisors in common. If, to the contrary, p were a prime divisor of both $m^2 + n^2$ and $2mn$, then p would be odd, since $m^2 + n^2 \equiv 1(\text{mod}2)$, by virtue of the hypothesis $m + n \equiv 1(\text{mod}2)$. Thus, $p|2mn$ implies, since $(p, 2) = 1$, that $p|mn$

(by Lemma 1). But p is a prime, so $p|mn$ implies that p must divide at least one of m, n. If $p|m$, then from $p|m^2+n^2$, it follows that $p|n^2$, and so $p|n$. Thus, $p|m$ and $p|n$ contradicting the hypothesis that $(m,n)-1$

(ii) A similar argument left to the reader (p must divide the sum of m^2+n^2 and m^2-n^2, and their difference. Hence, $p|2n^2$ and $p|2m^2$, which since p is odd, eventually implies $p|n$ and $p|m$, a contradiction).

(iii) A similar argument as in (i).

□

3 A theorem and a corollary

Theorem 1. *Suppose that ABC is a Pythagorean triangle with the right angle at C; and with the three sidelengths satisfying the formulas in (1), namely $a = d(m^2-n^2),\ b = d(2mn),\ c = d(m^2+n^2)$, where d, m, n are positive integers such that $m > n,\ (m,n) = 1$, and $m+n \equiv 1 (\text{mod } 2)$.*

Let P be a point on the hypotenuse $\overline{AB}$, distinct from A and B. Furthermore, suppose that D is the foot of the perpendicular from P to the side $\overline{CB}$; and E the foot of the perpendicular from P to the side $\overline{CA}$, as in Figure 1. Then, either both right triangles BDP and APE are Pythagorean or neither of them is.

Moreover, if they are both Pythagorean, then the sidelengths $|\overline{BD}| = a-y$, $|\overline{DP}| = x$, and $|\overline{BP}| = h_1$ of the triangle BDP satisfy the formulas,

$$a-y = \delta(m^2-n^2),\ x = \delta(2mn),\ h_1 = \delta(m^2+n^2).$$

While the sidelengths $|\overline{PE}| = y$, $|\overline{EA}| = b-x$, $|\overline{PA}| = h_2$ of the triangle PEA satisfy the formulas

$$y = (d-\delta)(m^2-n^2),\ b-x = (d-\delta)(2mn),\ h_2 = (d-\delta)(m^2+n^2)$$

where δ is a positive integer such that $1 \leq \delta \leq d-1$

Proof. Suppose that the triangle BDP is Pythagorean. We will prove that the tringle APE must also be Pythagorean; then so must the triangle BDP be.

Since the triangle BDP is Pythagorean, its three sidelengths, x, $a-y$, and h_1 (see Figure 1) must be natural numbers. From (3i)

$$\begin{aligned} \Rightarrow \quad x &= \frac{b \cdot h_1}{c} \underset{\text{by (1)}}{=} \frac{d(2mn)}{d(m^2+n^2)} \cdot h_1; \\ x &= \frac{2mnh_1}{m^2+n^2} \end{aligned} \tag{4}$$

From the conditions $(m, n) = 1$ and $m + n \equiv 1 (\text{mod } 2)$, it follows by Lemma 2(i) that

$$(m^2 + n^2, 2mn) = 1 \tag{4i}$$

Since x is a natural number, equation (4) says that the integer $m^2 + n^2$ must be a divisor of the product $2mnh_1$, which clearly implies, by (4i) and Lemma 1, that h_1 must be divisible by $m^2 + n^2$.

$$h_1 = \delta \cdot (m^2 + n^2) \tag{4ii}$$

for some positive integer δ; and since h_1 is the length hypotenuse $\overline{BP}$ (triangle BDP), and the point P lies strictly between A and B, it is clear that

$$h_1 = \left|\overline{BP}\right| < c = \left|\overline{BA}\right| = d(m^2 + n^2),$$

which together with (4ii) clearly show that

$$\begin{aligned} &1 \leq \delta < d; \text{ or equivalently,} \\ &1 \leq \delta \leq d - 1 \end{aligned} \tag{4iii}$$

Note that by (4iii), we must have $d \geq 2$. Going back to (4) and using (4ii) we get

$$x = (2mn)\delta \tag{4iv}$$

and so, by (4iv), (3i), (4ii), and (1), we further obtain

$$\begin{aligned} a - y &= \delta(m^2 - n^2); \; y = a - \delta(m^2 - n^2); \\ y &= d(m^2 - n^2) - \delta(m^2 - n^2) = (d - \delta)(m^2 - n^2) \end{aligned} \tag{4v}$$

By using (1), (3i), and (4v) we also get

$$b - x = (d - \delta)(2mn)$$

and

$$h_2 = (d - \delta)(m^2 + n^2).$$

The proof is complete. $\square$

Theorem 2. *Let CBA be a Pythagorean triangle, with the 90 degree angle at C. Also, let $\left|\overline{CB}\right| = a$, $\left|\overline{CA}\right| = b$ and $\left|\overline{BA}\right| = c$, be the three sidelengths so that $a = d(m^2 - n^2)$, $b = d(2mn), c = d(m^2 + n^2)$ where m, n, d are positive integers such that $m > n$, $(m, n) = 1$, and $m + n \equiv 1(\text{mod } 2)$.*

Let P be a point on the hypotenuse $\left|\overline{BA}\right|$ and strictly between the end-points B and A.

Let D and E be the feet of the perpendiculars from the point P to the sides $\overline{CB}$ and $\overline{CA}$ respectively.

(i) If $d = 1$. i.e., if the Pythagorean triangle CBA is primitive, then neither of the right triangles PDB and PEA is Pythagorean.

(ii) If $d = 2$, and the point P is coincident with the midpoint M of the hypotenuse $\overline{BA}$, then both triangles PDB and PEA are Pythagorean. Otherwise, if $P \neq M$, neither of these two triangles is Pythagorean.

(iii) If $d = 3$, and the point P is such that $\dfrac{\left|\overline{PB}\right|}{\left|\overline{PA}\right|} = \dfrac{1}{3}$ or $\dfrac{2}{3}$, then both triangles PDB and PEA are Pythagorean. Otherwise, if $\dfrac{\left|\overline{PB}\right|}{\left|\overline{PA}\right|} \neq \dfrac{1}{3}, \dfrac{2}{3}$, then neither of these triangles are Pythagorean.

Proof. (i) If $d = 1$, then neither of the two right triangles, BDP and PEA can be Pythagorean since according to Theorem 1, the natural number δ must satisfy $1 \leq \delta \leq d - 1$, which is impossible when $d = 1$.

(ii) Suppose that $d = 2$.

If the point P coincides with the midpoint M of the hypotenuse $\overline{BA}$, then each of the triangles BDP and PEA is half the size of the triangle CBA. So, by inspection,

$$
\begin{aligned}
|\overline{BD}| = |\overline{PE}| &= \frac{a}{2} = \frac{2(m^2-n^2)}{2} = m^2-n^2 \\
|\overline{DP}| = |\overline{EA}| &= \frac{b}{2} = \frac{2(2mn)}{2} = 2mn \\
|\overline{BP}| = |\overline{PA}| &= \frac{c}{2} = \frac{2(m^2+n^2)}{2} = m^2+n^2,
\end{aligned}
$$

which proves that both triangles BDP and PEA are (in fact primitive) Pythagorean triangles. Conversely, if both triangles are Pythagorean, then by Theorem 1, it follows that $1 \le \delta \le d-1 = 2-1 = 1$, $1 \le \delta \le 1$, $\delta = 1$ which establishes that each of the triangles is half the size of triangle of CBA; which implies that P is the midpoint of $\overline{BA}$.

(iii) Assume that $d = 3$.

Suppose $\frac{|\overline{PB}|}{|\overline{PA}|} = \frac{1}{3}$ or $\frac{2}{3}$. If $\frac{|\overline{PB}|}{|\overline{PA}|} = \frac{1}{3}$, then the triangle BDP is $\frac{1}{3}$ the size of triangle CBA and the triangle PEA is $\frac{2}{3}$ the size of CBA. We have,

$$
\begin{aligned}
|\overline{BD}| &= \frac{a}{3} = \frac{3(m^2-n^2)}{3} = m^2-n^2, \quad |\overline{DP}| = \frac{b}{3} = \frac{3(2mn)}{3} = 2mn, \\
|\overline{PB}| &= \frac{c}{3} = \frac{3(m^2+n^2)}{3} = m^2+n^2
\end{aligned}
$$

and $|\overline{PE}| = \frac{2a}{3} = 2(m^2-n^2)$, $|\overline{EA}| = \frac{2b}{3} = 2(2mn)$, $|\overline{PA}| = \frac{2c}{3} = 2(m^2+n^2)$. It is clear that both triangles BDE and PEA are Pythagorean.

The argument for the case $\frac{|\overline{PB}|}{|\overline{PA}|} = \frac{2}{3}$ is similar (we omit the details).

Now, the converse. Assume that both triangles, BDP and PEA, are Pythagorean. Then by Theorem 1 we must have,

$$1 \le \delta \le d-1 = 3-1 = 2; \quad \delta = 1 \text{ or } 2.$$

Using the formulas for the sidelengths (of triangles BDP and PEA), found in Theorem 1 we easily see that $\dfrac{|\overline{PA}|}{|\overline{PB}|} = \dfrac{1}{3}$, if $\delta = 1$. While $\dfrac{|\overline{PA}|}{|\overline{PB}|} = \dfrac{2}{3}$, if $\delta = 2$. The proof is complete.

$\square$

4 Three special cases

A. **Case 1: When the point P is the midpoint M of the hypotenuse $\overline{BA}$**

By inspection, it is clear that all six right triangles BDP, PEA, CDP, EPD, DCE, and PEC are all congruent and each of them is half the size of triangle BCA. Clearly then, by (1), these six triangles will be Pythagorean if and only if the integer d in (1) is even.

Theorem 3. *Let BCA be a Pythagorean triangle with the 90 degree angle at C and $|\overline{CB}| = a = d(m^2 - n^2)$, $|\overline{CA}| = b = d(2mn)$, $|\overline{BA}| = c = d(m^2 + n^2)$, where d, m, n are postive integers such that $m > n$, $(m,n) = 1$ and $m + n \equiv 1 (\text{mod } 2)$.*

Let M be the midpoint of the hypotenuse $\overline{BA}$ and D, E the feet of the perpendiculars from M to the sides $\overline{CB}$ and $\overline{CA}$ respectively (so D and E are the midpoints of $\overline{CB}$ and $\overline{CA}$). Then, the six right angles, BDM, MEA, CDM, EMD, DCE, and MEC are congruent and have sidelengths as follows:

$$\begin{aligned} &\textit{length of horizontal side} && = \frac{d}{2}(2mn) = dmn \\ &\textit{length of veritical side} && = \frac{d(m^2 - n^2)}{2} \\ &\textit{length of hypotenuse} && = \frac{d(m^2 + n^2)}{2} \end{aligned}$$

If d is an even natural number, then the above six triangles are Pythagorean; otherwise, if d is odd, they are non-Pythagorean.

B. **Case 2: When the point P is the foot I of the angle bisector of the 90° angle at C**

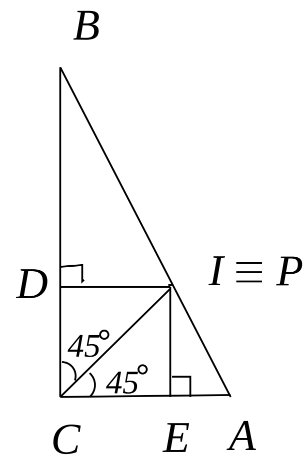

Figure 2

Using the notation of Theorem 1, we have $\left|\overline{BD}\right| = a - y, \left|\overline{DI}\right| = x,$

$\left|\overline{BP}\right| = h_1,\ \left|\overline{EA}\right| = b - x,$

$\left|\overline{EI}\right| = y,$ and $\left|\overline{IA}\right| = h_2.$

Clearly, we have $x = y$ in this case. Note that the four congruent isosceles right triangles DCI, IEC, DCE, DIE cannot be Pythagorean (no Pythagorean triangle is isosceles).

By Theorem 1, the two right triangles BDI and IEA are either both Pythagorean or neither of them are. If they are both Pythagorean, then by Theorem 1 we have, in particular, $x = \delta(2mn)$ and $y = (d-\delta)(m^2 - n^2)$ with m, n, d, δ being positive integers such that $m > n,\ (m, n) = 1,\ m + n \equiv 1 (\mathrm{mod}\ 2)$ and $1 \leq \delta \leq d - 1$ (and so $d \geq 2$).

Since $x = y$, we must have

$$\delta(2mn) = (d-\delta)(m^2 - n^2) \tag{5}$$

By Lemma 2(iii), we know that $(m^2 - n^2, 2mn) = 1$. So, by Lemma 1 and (5) it follows that $2mn | d - \delta$ and $m^2 - n^2 | \delta$ which, in turn, leads to (when we go back to (5))

$$\left\{\begin{array}{l} \delta = K \cdot (m^2 - n^2) \\ \\ d - \delta = K \cdot (2mn), \\ \\ \text{for some positive integer } K. \\ \\ \text{Hence } d = k \cdot (m^2 - n^2 + 2mn) \end{array}\right\} \tag{5i}$$

Note that clearly, from (5i), $1 \leq \delta \leq d-1$. In fact, the smallest possible value of d is 7; obtained for $K = 1$ and $m = 2, n = 1$. Moreover, $1 \leq \delta \leq d - 4$ since the smallest possible value of $K \cdot (2mn)$ is 4.

Using (5i) and Theorem 1, one can compute in terms of m, n, and K. The other four sidelengths of the triangles BDI and IEA. Also, by (5i) we get $x = \delta(2mn) = K(2mn)(m^2 - n^2) = y$. We now state the following theorem.

Theorem 4. *Let CBA be a Pythagorean triangle with the 90° angle at C and sidelengths given by*

$$\left|\overline{CB}\right| = a = d(m^2 - n^2),\ \left|\overline{CA}\right| = b = d(2mn),\ \left|\overline{BA}\right| = c = d(m^2 + n^2),$$

where d, m, n are positive integers such that, $m > n,\ m+n \equiv 1 (\text{mod } 2)$, and $(m, n) = 1$. Let I be the foot of the perpendicular of the angle bisector (of the 90° angle at C) to the hypotenuse $\overline{BA}$.

Also, let D and E be the feet of the perpendiculars from the point I to the sides $\overline{CB}$ and $\overline{CA}$ respectively. Then, the two right triangles BDI and IEA are both Pythagorean precisely when (i.e., if and only if), $d = K \cdot (m^2 - n^2 + 2mn)$ for some integer K. If $d = K \cdot (m^2 - n^2 + 2mn)$, then the sidelengths of triangle BDI are given by $\left|\overline{DI}\right| = x = K \cdot (2mn)(m^2 - n^2)$, $\quad h_1 = \left|\overline{BI}\right| = K(m^2 - n^2)(m^2 + n^2) = K(m^4 - n^4)$, and $\left|\overline{BD}\right| = a - y = K \cdot (m^2 - n^2)^2$ and the sidelengths of triangle IEA are given by

$$\left|\overline{IE}\right| = y = K(2mn)(m^2 - n^2),$$

$$\left|\overline{EA}\right| = b - x = K(2mn)(2mn) = K \cdot (2mn)^2,$$

and $h_2 = \left|\overline{IA}\right| = K \cdot (2mn)(m^2 + n^2)$.

If the integer d *is not divisible by* $m^2 - n^2 + 2mn$*, then neither of the triangles,* BDI *and* IEA*, is Pythagorean.*

C. **Case 3: When the point P is the foot F of the perpendicular from the vertex C to the hypotenuse $\overline{BA}$**

In this part, instead of using Theorem 1, we will first compute the sidelengths of the triangles BDF, FEA, and the four congruent triangles FDC, DFE, DCE, and CFE in terms of (the sidelengths) a, b, c. After that we will implement the formulas in (1) in order to express the above sidelengths in terms of the integers d, m, n.

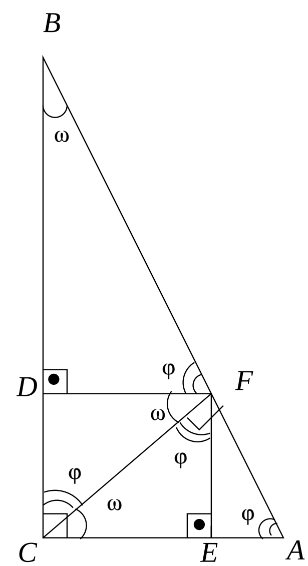

Figure 3

After that we will implement Lemma 2 to be able to draw the conclusions which will lead to Theorem 5. Note that since F is the foot of the perpendicular from C to the hypotenuse $\overline{BA}$, the aforementioned six right triangles are all similr to the triangle CBA. Let ω and φ be the degree measures of the angles $\angle CBA$ and $\angle CAB$ respectively (see Figure 3).

We have (and we set)

$$\left\{\begin{array}{l} \left|\overline{CB}\right| = a,\ \left|\overline{CA}\right| = b,\ \left|\overline{BA}\right| = c \\ \\ \left|\overline{DF}\right| = \left|\overline{CE}\right| = x,\ \left|\overline{EA}\right| = b - x \\ \\ \left|\overline{DC}\right| = \left|\overline{FE}\right| = y,\ \left|\overline{BD}\right| = a - y \\ \\ \left|\overline{BF}\right| = h_1,\ \left|\overline{FA}\right| = h_2,\ \left|\overline{CF}\right| = \left|\overline{DE}\right| = h \end{array}\right\} \tag{6}$$

Furthermore,

$$\sin\omega = \frac{y}{h} = \cos\varphi = \frac{b}{c} \text{ and } \cos\omega = \frac{h}{b} = \frac{a}{c};$$

and thus $h = \dfrac{ab}{c}$, which implies $y = h \cdot \cos\varphi = h \cdot \dfrac{b}{c} = \dfrac{ab}{c} \cdot \dfrac{b}{c} = \dfrac{ab^2}{c^2}$. So, $a - y = a - \dfrac{ab^2}{c^2} = \dfrac{a(c^2 - b^2)}{c^2} =$ (since $c^2 = a^2 + b^2$) $\dfrac{a \cdot a^2}{c^2}$; $a - y = \dfrac{a^3}{c^2}$
Next we calculate the lengths x and $b - x$. We have $\tan\omega = \cot\varphi = \dfrac{y}{x}$; $\cot\omega = \tan\varphi = \dfrac{x}{y}$, and $\tan\varphi = \dfrac{a}{b}$ which gives $\dfrac{x}{y} = \dfrac{a}{b}, x = \dfrac{a}{b} \cdot y$. Since $y = \dfrac{ab^2}{c^2}$ (see above), we obtain $x = \dfrac{a}{b} \cdot \dfrac{ab^2}{c^2} = \dfrac{ba^2}{c^2}$. From this we get $b - x = b - \dfrac{ba^2}{c^2} = \dfrac{b(c^2 - a^2)}{c^2} = \dfrac{b \cdot b^2}{c^2} = \dfrac{b^3}{c^2}$, since $c^2 - a^2 = b^2$.
Also, $\sin\omega = \dfrac{x}{h_1}$; $h_1 = \dfrac{1}{\sin\omega} \cdot x = \dfrac{c}{b} \cdot \dfrac{ba^2}{c^2} = \dfrac{a^2}{c}$. Similarly, we have $\sin\varphi = \dfrac{y}{h_2}$; $h_2 = \dfrac{1}{\sin\varphi} \cdot y = \dfrac{c}{a} \cdot \dfrac{ab^2}{c^2} = \dfrac{b^2}{c}$. We summarize these lengths as follows:

Sidelengths of triangle BDF

$$\left(\left|\overline{BD}\right| = a - y = \frac{a^3}{c^2},\ \left|\overline{DF}\right| = x = \frac{ba^2}{c^2},\ \left|\overline{BF}\right| = h_1 = \frac{a^2}{c}\right) \tag{6i}$$

Sidelengths of triangle FEA

$$\left(|\overline{FE}| = y = \frac{ab^2}{c^2},\ |\overline{EA}| = b - x = \frac{b^3}{c^2},\ |\overline{FA}| = h_2 = \frac{b^2}{c}\right) \tag{6ii}$$

Sidelengths of the four congruent triangles $FDC,\ DFE,\ DCE,\ CFE$

$$\begin{aligned}(|\overline{DC}| &= |\overline{FE}| = y = \frac{ab^2}{c^2},\ |\overline{DF}| = |\overline{CE}| \\ &= x = \frac{ba^2}{c^2},\ |\overline{CF}| = |\overline{DE}| = \frac{ab}{c} = h\Big)\end{aligned} \tag{6iii}$$

Next, we combine the length formulas in (6i), (6ii), and (6iii) with the formulas in (1), since CBA is a Pythagorean triangle, to obtain the following.

$$\left\{\begin{aligned} a - y &= \frac{d\cdot(m^2-n^2)^3}{(m^2+n^2)^2},\ x = \frac{d\cdot(m^2-n^2)^2\cdot(2mn)}{(m^2+n^2)^2} \\ y &= \frac{d\cdot(m^2-n^2)\cdot(2mn)^2}{(m^2+n^2)^2},\ b - x = \frac{d\cdot(2mn)^3}{(m^2+n^2)^2} \\ h_1 &= \frac{d\cdot(m^2-n^2)^2}{m^2+n^2},\ h_2 = \frac{d\cdot(2mn)^2}{m^2+n^2} \\ h &= \frac{d\cdot(2mn)\cdot(m^2-n^2)}{m^2+n^2} \end{aligned}\right\} \tag{7}$$

The following lemma from number theory is well-known and comes in handy.

Lemma 3. *Let $i_1,\ i_2,\ i_3,\ e_1,\ e_2,\ e_3$ be positive integers such that $(i_1, i_2) = 1 = (i_1, i_3)$. Then,*

(a) $(i_1^{e_1},\ i_2^{e_2}) = 1$

(b) $(i_1^{e_1},\ i_2^{e_2}\cdot i_3^{e_3}) = 1$

It follows from Lemmas 2 and 3 that

$$\left\{\begin{array}{l} \left((m^2+n^2)^2,\ (m^2-n^2)^3\right)=1, \\[2ex] \left((m^2+n^2)^2,\ (2mn)^2\right)=1 \\[2ex] \left(m^2+n^2,\ (m^2-n^2)^2\right)=1, \\[2ex] \left(m^2+n^2,\ (2mn)^2\right)=1 \\[2ex] (m^2+n^2,\ (2mn)\cdot(m^2-n^2))=1 \\[2ex] \left((m^2+n^2)^2,\ (m^2-n^2)^2\cdot(2mn)\right)=1 \\[2ex] \left((m^2+n^2)^2,\ (m^2-n^2)\cdot(2mn)^2\right)=1 \end{array}\right\} \tag{8}$$

A careful look at formulas (7) and the coprimeness conditions in (8), in conjunction with Lemma 1, reveals that either all six triangles, BDF, FEA, FDC, DFE, DCE, and CFE are Pythagorean; or none of them are.

They are all Pythagorean precisely (i.e., if and only if) the integer d is divisible by $(m^2+n^2)^2$, i.e., when

$$\left\{\begin{array}{l} d=K\cdot(m^2+n^2)^2 \\[2ex] \text{for some positive integer } K \end{array}\right\} \tag{9}$$

This is precisely when all seven numbers $y,\ a-y,\ x,\ b-x,\ h_1,\ h_2,$ and h are integers. When (9) holds true, we can compute, via (7) and (61), (6ii), and (6iii) all the sidelengths in terms of the integers $m, n,$ and K.

We have the following theorem.

Theorem 5. *Let CBA be a Pythagorean triangle, with the 90-degree angle at C. With $\left|\overline{CB}\right|=a=d(m^2-n^2)$, $\left|\overline{CA}\right|=b=d(2mn)$, $\left|\overline{BA}\right|=$*

$d(m^2+n^2)=c$, where d,m,n are positive integers such that $m>n$, $(m,n)=1$, and $m+n\equiv 1(\mathrm{mod}2)$.

Also, let F be the foot of the perpendicular from the vertex C to the hypotenuse $\overline{BA}$. Then, the six similar triangles BDF, FEA, (and the four congruent ones) FDC, DFE, DCE, CFE are either all Pythagorean or none of them are. They are all Pythagorean precisely when (i.e., if and only if) $d=K\cdot(m^2+n^2)^2$, for some positive integer K. When d satisfies the said condition, the sidelengths of the above six triangles are given by the following formulas.

For triangle BDF

$\left|\overline{BD}\right| = a-y = K\cdot(m^2-n^2)$, $\left|\overline{DF}\right| = x = K\cdot(m^2-n^2)^2\cdot(2mn)$, and $h_1=K\cdot(m^2+n^2)\cdot(m^2-n^2)^2$

For triangle FEA:

$\left|\overline{FE}\right| = y = K\cdot(m^2-n^2)\cdot(2mn)^2$, $\left|\overline{EA}\right| = b-x = K\cdot(2mn)^3$, *and* $h_2=K\cdot(m^2+n^2)\cdot(2mn)^2$.

For the four congruent triangles FDC, DFE, DCE, CFE:

$\left|\overline{DC}\right| = \left|\overline{FE}\right| = y = K\cdot(m^2-n^2)\cdot(2mn)^2$,

$\left|\overline{DF}\right| = \left|\overline{CE}\right| = x = K\cdot(2mn)\cdot(m^2-n^2)^2$,

and

$h=\left|\overline{CF}\right| = \left|\overline{DE}\right| = K\cdot(2mn)\cdot(m^2-n^2)\cdot(m^2+n^2) = K\cdot(2mn)\,(m^4-n^4)$.

Numerical Examples

If we take $K=1$ and $mn\leq 4$, then $K=1$ and $m=2$, $n=1$; or $K=1$ and $m=4, n=1$.

(a) $K=1$, $m=2$, $n=1$. We obtain the following:

$$d = 1 \cdot (2^2 + 1^2)^2 = 5^2 = 25,\ h = 60,\ h_1 = 45,\ h_2 = 80,$$

$$y = 48,\ a - y = 75 - 48 = 27,\ x = 36,\ b - x = 100 - 36 = 64,$$

$$a = 75,\ b = 100,\ c = 125$$

(b) $K = 1,\ m = 4,\ n = 1$. We have the following:

$$d = 289,\ a - y = 15,\ x = 1800,\ h_1 = 3825,$$

$$y = 960,\ b - x = 512,\ h_2 = 1088,\ h = 1404$$

$$a = 4335,\ b = 2312,\ c = 4913$$

5 Exactly $(d-1)$ positions of P

Given a Pythagorean triangle CBA, as in Figure 1, and with the point P on the hypotenuse $\overline{BA}$, and D and E being the perpendicular projections of P on the sides $\overline{CB}$ and $\overline{CA}$ respectively. We know from Theorem 1 that either both triangles BDP and PEA are Pythagorean, or neither of them are. The integer δ, as described in Theorem 1 must satisfy $1 \leq \delta \leq d-1$; which means that $d \geq 2$ is a necessary condition. There are $(d-1)$ choices for δ. If we subdivide the hypotenuse $\overline{BA}$ into d equal length segments, each segment having length $m^2 + n^2$, it is easily seen that for each such position of the point P both triangles BDP and PEA are Pythagorean. There are exactly $(d-1)$ such positions for the point P along the hypotenuse $\overline{BA}$. These are the points $P_1, \ldots, P_{d-1}$; so that each of the consecutive line segments $\overline{BP_1}, \overline{P_1P_2}, \ldots, \overline{P_{d-1}A}$ (exactly d line segments) has length $m^2 + n^2$.

We postulate the following theorem.

Theorem 6. *Let CBA be a Pythagorean triangle with the $90°$ angle at the vertex C. With sidelengths given by $\left|\overline{CB}\right| = a = d(m^2 - n^2)$, $\left|\overline{CA}\right| = b = d(2mn)$, $\left|\overline{BA}\right| = d(m^2 + n^2)$, where d, m, n are positive integers such that $d \geq 2,\ m > n,\ (m, n) = 1$, and $m + n \equiv \pmod 2$. Also, let $P_1, \ldots, P_{d-1}$ be the $(d-1)$ points on the hypotenuse $\overline{BA}$ such that the d consecutive line segments $\overline{BP_1},\ \overline{P_1P_2}, \ldots, \overline{P_{d-1}A}$ have equal lengths; each having length $m^2 + n^2$. Then there are exactly $(d-1)$ points P on the hypotenuse $\overline{BA}$ such that both trangles BDP and PEA are Pythagorean where D and*

E are the feet of the perpendiculars from P to the sides $\overline{CB}$ and $\overline{CA}$ respectively. These $(d-1)$ points are precisely the points $P_1, \ldots, P_{d-1}$ described above. Furthermore, each pair of Pythagorean triangles BD_iP_i and P_iE_iA have sidelengths given by $\left|\overline{BD_i}\right| = i \cdot (m^2 - n^2)$, $\left|\overline{D_iP_i}\right| = i(2mn)$, $\left|\overline{BP_i}\right| = i\,(m^2 + n^2)$, $\left|\overline{P_iE_i}\right| = (d-i)\,(m^2 - n^2)$, $\left|\overline{E_iA}\right| = (d-i)(2mn)$, $\left|\overline{P_iA}\right| = (d-i)\,(m^2+n^2)$, for $i = 1, \ldots, d-1$; and where D_i and E_i are the perpendicular projections of the point P_i onto the sides $\overline{CB}$ and $\overline{CA}$ respectively.

6 Other cases

In this section, we explore the following question. If in addition to the two triangles in Figure 1, BDP and PEA being Pythagorean, we require that the four congruent triangles DCE, PEC, CDP, EPD, also be Pythagorean. What are the necessary and sufficient conditions for this to occur?

For these four congruent triangles to be Pythagorean, the integers $x = \left|\overline{DP}\right| = \left|\overline{CE}\right|$ and $y = \left|\overline{DC}\right| = \left|\overline{PE}\right|$ must satisfy the condition,

$$x^2 + y^2 = \text{ perfect square.}$$

Combining this with Theorem 6 leads to the following theorem.

Theorem 7. *Let CBA be a Pythagorean triangle with the 90 degree angle at the vertex C; and with sidelengths, $a = \left|\overline{CB}\right| = d\,(m^2 - n^2)$, $b = \left|\overline{CA}\right| = d(2mn)$, $c = \left|\overline{BA}\right| = d\,(m^2+n^2)$ where d, m, n are positive integers such that $d \geq 2$, $m > n$, $(m,n) = 1$, and $m + n \equiv 1 (\mathrm{mod}\ 2)$. Let P be a point on the hypotenuse $\overline{BA}$, and D and E be the feet of the perpendiculars from the point P onto the sides $\overline{CB}$ and $\overline{CA}$ respectively. Also, let $x = \left|\overline{DP}\right| = \left|\overline{CD}\right|$, $y = \left|\overline{DC}\right| = \left|\overline{PE}\right|$ so that*

$$a - y = \left|\overline{BD}\right| \text{ and } b - x = \left|\overline{EA}\right|.$$

Then, the two right triangles BDP and PEA, as well as the four congruent triangles, DCE, PEC, CDP, EPD, are all (six triangles) are Pythagorean if and only if there exist positive integers D, M, N such that

$$M > N,\ (M,N) = 1,\ M + N \equiv 1 (\mathrm{mod}\ 2)$$

and with either

$$\left.\begin{cases} y = \delta\,(m^2 - n^2) = D \cdot (M^2 - N^2) \\[2ex] x = (d - \delta) \cdot (2mn) = D \cdot (2MN) \\[2ex] \delta \text{ a positive integer such that } 1 \le \delta \le d - 1 \end{cases}\right\} \tag{10i}$$

or

$$\left.\begin{cases} y = \delta\,(m^2 - n^2) = D \cdot (2MN) \\[2ex] x = (d - \delta)(2mn) = D \cdot (M^2 - N^2) \\[2ex] \delta \text{ a positive integer such that } 1 \le \delta \le d - 1 \end{cases}\right\} \tag{10ii}$$

The following example shows that there exist nonprimitive Pythagorean triangles such that there is no point P on the hypotenuse $\overline{BA}$ such that all six triangles BDP, PEA, DCE, PEC, CDP, EPD, are Pythagorean.

Example: Take $d = 5,\ m = 2,\ n = 1$. Then the sidelengths of triangle CBA are $a = 5 \cdot (2^2 - 1^2) = 15,\ b = 5 \cdot (2 \cdot 2 \cdot 1) = 20$, and $c = 5 \cdot (2^2 + 1^1) = 25$. The possible values of the integer δ are $\delta = 1, 2, \ldots, d - 1 = 1, 2, 3, 4$. Using the formulas $y = \delta\,(m^2 - n^2)$ and $x = (d - \delta)(2mn)$ we have the following.

1. $\delta = 1:\ y = 3,\ x = (5 - 1) \cdot 4 = 16$
   and $y^2 + x^2 = 9 + 256 = 265$ not an integer square.
2. $\delta = 2,\ y = 2 \cdot 3 = 6,\ x = (5 - 2) \cdot 4 = 12$
   and $y^2 + x^2 = 36 + 144 = 180$, not a perfect square.
3. $\delta = 3,\ y = 3 \cdot 3 = 9,\ x = (5 - 3) \cdot 4 = 8$
   and $y^2 + x^2 = 81 + 64 = 145$, not an integer square.
4. $\delta = 4,\ y = 4 \cdot 3 = 12,\ x = (5 - 4) \cdot 4 = 4$
   and $y^2 + x^2 = 144 + 16 = 160$, not a perfect square.

There are many ways in which one can use the conditions (10i) or (10ii) of Theorem 7 in order to produce families of Pythagorean triangles such that each member (of those families) has the property that there is a point P on

its hypotenuse such that all six triangles (as described in Theorem 7) are Pythagorean. We produce such a family.

Family 1: Consider (10i):

$$\left.\begin{aligned} y &= \delta\left(m^2-n^2\right) = D\left(M^2-N^2\right) \\ x &= (d-\delta)(2mn) = D(2MN) \end{aligned}\right\} \tag{10i}$$

Let K be a positive integer.
Take $D = K \cdot mn\left(m^2-n^2\right)$.

From the second equation in (10i) we obtain

$$d-\delta = K \cdot MN\left(m^2-n^2\right)$$

and from the first equation in (10i) we get

$$\delta = Kmn\left(M^2-N^2\right).$$

Hence, $d = \delta + KMN\left(m^2-n^2\right) = K \cdot \left[mn\left(M^2-N^2\right) + MN\left(m^2-n^2\right)\right]$. Obviously $1 \leq \delta \leq d-1$ and $d \geq 2$, as required. We have the following.

Family 1

Let m, n, M, N be positive integrs such that $m > n$, $(m,n) = 1$, $m+n \equiv (\mathrm{mod}\ 2)$, $M > N$, $(M,N) = 1$ $M+N \equiv 1(\mathrm{mod}\ 2)$. Also, let K be a positive integer and $\delta = Kmn\left(M^2-N^2\right)$, $d = K\cdot\left[mn\left(M^2-N^2\right) + MN\left(m^2-n^2\right)\right]$. Consider the Pythagorean triangle CBA with sidelengths $\left|\overline{CB}\right| = a = d\left(m^2-n^2\right)$, $\left|\overline{CA}\right| = b = d(2mn)$, $\left|\overline{BA}\right| = c = d\left(m^2+n^2\right)$. Let P be the point on the hypotenuse $\left|\overline{BA}\right|$ such that $\left|\overline{BP}\right| = h_1 = \delta\left(m^2+n^2\right)$; and let D and E be the perpendicular projections of P onto the sides $\overline{CB}$ and $\overline{CA}$ respectively. Then all six right triangles BDP, PEA, DCE, PEC, CDP and EPD are Pythagorean.

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For Pythagorean triangles see pp. 510-515.

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